An Analysis of the Ibovespa Index Using GARCH Models with Normal and Non-normal Distributions

Luiz Guilherme Soré Spricido
Actuary, graduated at the University of São Paulo with a Master degree at the Cass Business School – City, University of London. He has worked in actuarial, investment and financial consulting areas, spending much of his career at Itaú Unibanco. He has great interest in quantitative risk management, machine learning and in the general development of Actuarial Science in the Brazil.
luiz.spricido@gmail.com

Summary

The aim of this paper is to verify how different models of asset returns can affect the risk measures calculated by a risk manager as they become consistently more important in an increasingly complex financial context.

Firstly, this paper briefly analyses the unconditional distribution for the daily log returns on Brazil’s Ibovespa stock market index. Some key statistics (mean, standard deviation, etc.) are applied to that financial time series and the most commonly used assumption of normally distributed returns is verified using well-known statistical tools, such as numerical normality tests and Q-Q plots. In addition, a Ljung-Box test is conducted to verify the assumption of randomness in the data.

Secondly, a GARCH(1, 1) model with leverage is fitted to the data due to the presence of conditional heteroskedasticity in that time series. Here, it is first assumed that the innovations follow a normal distribution and the goodness of fit is evaluated using a Q-Q plot. Due to the poor fit of the Gaussian distribution on the left tail, the Student’s t and the Generalised Pareto distributions are fitted to the time series as an attempt to get a better fit, and their respective Q-Q plots are shown.

Finally, to evaluate the impact of these different assumptions from a risk management perspective, the daily Value-at-Risk (VaR) and the Expected Shortfall measures are calculated for the period from January 2017 through April 2017 using all three conditional distributions.

Key Words

Asset returns, risk, index Ibovespa, Value-at-Risk (VaR), Expected Shortfall (ES).

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An Analysis of the Ibovespa Index Using GARCH Models with Normal and Non-normal Distributions

Sinopse
Uma análise do índice Ibovespa usando modelos GARCH com distribuições normal e não normal

Luiz Guilherme Soré Spricido
Atuário graduado pela Universidade de São Paulo com mestrado pela Cass Business School – City, University of London. Durante sua carreira trabalhou em áreas atuariais, de investimentos e em consultoria financeira, passando grande parte de sua carreira no Itaú Unibanco. Possui grande interesse pelas áreas de gerenciamento de risco quantitativo, machine learning e pelo desenvolvimento geral da Ciência Atuarial no país.
luis.spricido@gmail.com

Resumo
O objetivo deste artigo é verificar como diferentes modelos de retorno de ativos podem afetar as medidas de risco calculadas por um gestor de risco à medida que se tornam consistentemente mais importantes em um contexto financeiro cada vez mais complexo.

Primeiramente, o artigo analisa brevemente a distribuição incondicional para os retornos diários do índice Ibovespa no Brasil. Algumas estatísticas chave (média, desvio padrão, etc.) são aplicadas a essa série temporal financeira e a suposição mais comumente usada de retornos normalmente distribuídos é verificada usando ferramentas estatísticas conhecidas, como testes de normalidade numérica e gráficos Q-Q. Além disso, um teste Ljung-Box é realizado para verificar a suposição de aleatoriedade nos dados.

Em segundo lugar, um modelo GARCH (1, 1) com alavancagem é ajustado aos dados devido à presença de heteroscedasticidade condicional nessa série temporal. Aqui, supomos primeiro que as inovações seguem uma distribuição normal e a qualidade do ajuste é avaliada usando um gráfico Q-Q. Devido ao fraco ajuste da distribuição Gaussiana na cauda esquerda, as distribuições Student e Pareto Generalizada são ajustadas à série temporal como uma tentativa de obter melhor ajuste e os respectivos gráficos Q-Q são mostrados.

Finalmente, para avaliar o impacto dessas diferentes premissas do ponto de vista do gerenciamento de risco, as medidas diárias de Value-at-Risk (VaR) e Expected Shortfall (ES) são calculadas para o período de janeiro de 2017 a abril de 2017 usando as três distribuições condicionais.

Palavras-Chave
Retorno de ativos, risco, índice Ibovespa, Value-at-Risk (VaR), Expected Shortfall (ES).

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**Sinopsis**

Un análisis del índice Ibovespa utilizando modelos GARCH con distribuciones normales y no normales

**Luiz Guilherme Soré Spricido**
Graduación en la Universidad de São Paulo con una maestría en la Cass Business School – City, University of London. Trabajó en áreas actuariales, de inversiones y en consultoría financiera, pasando gran parte de su carrera en Itaú Unibanco. Tiene gran interés por las áreas de gestión de riesgo cuantitativo, machine learning y por el desarrollo general de la Ciencia Actuarial en Brasil.
luiz.spricido@gmail.com

**Resumen**

El objetivo de este artículo es verificar cómo diferentes modelos de retorno de activos pueden afectar las medidas de riesgo calculadas por un gestor de riesgo a medida que se vuelven consistentemente más importantes en un contexto financiero cada vez más complejo.

Primero, el artículo analiza brevemente la distribución incondicional para los retornos diarios del índice Ibovespa en Brasil. Algunas estadísticas clave (media, desviación estándar, etc.) se aplican a esta serie temporal financiera y la suposición más comúnmente usada de retornos normalmente distribuidos se comprueba mediante herramientas estadísticas conocidas, como pruebas de normalidad numérica y gráficos Q-Q. Además, una prueba Ljung-Box se realiza para verificar la suposición de aleatoriedad en los datos.

En segundo lugar, un modelo GARCH (1, 1) con apalancamiento se ajusta a los datos debido a la presencia de heteroscedasticidad condicional en esta serie temporal. Aquí, suponemos primero que las innovaciones siguen una distribución normal y la calidad del ajuste es evaluada usando un gráfico Q-Q. Debido al débil ajuste de la distribución Gaussiana en la cola izquierda, las distribuciones Student y Pareto Generalizada se ajustan a la serie temporal como un intento de obtener un mejor ajuste y los gráficos Q-Q se muestran.

Finalmente, para evaluar el impacto de estas diferentes premisas desde el punto de vista de la gestión de riesgos, las medidas diarias de Value-at-Risk (VaR) y Expected Shortfall (ES) se calculan para el período de enero de 2017 a abril de 2017 usando las tres distribuciones condicional.

**Palabras-Clave**

Retorno de activos, riesgo, índice Ibovespa, Value-at-Risk (VaR), Expected Shortfall (ES).

**Sumario**

1. Introducción. 2. Propiedades del índice Ibovespa. 3. Modelando los datos. 3.1 Modelo GARCH. 3.2 Distribución normal. 3.3 Distribución t de Student 3.4 Teoría del Valor Extremo. 4. Cálculo de las medidas de riesgo. 4.1 Valor en Riesgo (VaR). 4.2 Expected Shortfall (ES). 5. Conclusión. 6. Referencias bibliográficas.
1. Introduction

The use of the Gaussian distribution, or Normal distribution, as a model to describe equity returns was first questioned by the works of Mandelbrot (1963) and Fama (1965) and has been systematically challenged by empirical evidence during the past 50 years. Events such as the Black Monday and the 2007/2008 crisis, among several others in the past decades, led to a dramatic plunge in stock indices and raised doubts about the assumption of normality.

The blind belief in those models may strongly underestimate the probability of extreme returns, whereas, Cont (2001) argues that for longer time intervals (e.g. monthly) the normal assumption starts to be increasingly more likely. Nonetheless, the empirical evidence of so-called heavy-tailed distribution of daily asset returns are particularly important from a market risk management perspective. Professionals in this field are primarily concerned with the left tail behaviour of the distribution of returns and its impact on a portfolio of assets. This is because negative returns may jeopardize a company's financial position and, in extreme cases, its entire operation.

Statistical quantities, also known as risk measures, describing the conditional or unconditional loss distribution (i.e. – P&L) of the portfolio have been commonly used to evaluate the risk of a financial position. Examples of risk measures include the Value-at-Risk (VaR) and the Expected Shortfall (ES), which are applied, among other purposes, to calculate the amount of money a company should hold in order to provide coverage against unexpected future losses (McNeil et al., 2015 p. 61). The economic capital required by a financial institution has been given a special attention by regulators around the world by means of frameworks, such as Basel III, Solvency II, and a recently released legislation by the Superintendência de Seguros Privados (SUSEP) addressed to the Brazilian insurance market regulating the requirement of capital based on market risk.

In addition to the above, Cont (2001) highlights additional empirically observed properties, also called stylized empirical facts, which are very important for modelling the conditional and unconditional distribution of returns and, hence, extremely relevant to conduct an appropriate risk evaluation of a financial position. One of these properties is that financial time series is usually observed to have time-varying volatility and the variance shows a positive correlation with its own past data, thus suggesting the presence of volatility clusters (Curto et al., 2009).

In order to capture the volatility clustering of equity returns, the paper produced by Bollerslev (1986), which improved the work of Engle (1982), introduced the Generalised Autoregressive Conditional Heteroskedasticity (GARCH) models. Mittnik et al. (1998) argue that, although these models are suitable for modelling the conditional distribution of returns (i.e. conditioning on past data), they still rely on the assumption of normally distributed innovations, which may not be desirable. Therefore, several subsequent papers proposed other non-normal conditional distributions, such as Student's t (Bollerslev, 1987), the Laplace distribution (Granger and Ding, 1995), among others (Nelson, 1991; Liu and Brorsen, 1995; Mittnik et al., 1998).
This paper analyses the conditional and, to some extent, the unconditional distribution of the daily log returns on the Ibovespa index. The next section gives some statistical properties of the time series and evaluates its compatibility with the Normal distribution. The third section presents a description of the GARCH(1, 1) and its fit to the data, considering Gaussian and Student’s innovations. In the same section, an extreme value theory approach is given to the data and a Generalised Pareto Distribution (GPD) is fitted to both tails of the conditional distribution. The fourth section evaluates and discusses the daily VaR and ES using all the three different approaches described above. The final section presents a conclusion for this paper.

2. Properties of the Ibovespa index

The time series is composed of daily closing prices of the Ibovespa index, the main Brazilian stock index. The data covers the period from 16 November 2006 through 31 May 2017, resulting in 2,645 observations. The returns of those observations are calculated as follows:

\[ r_t = \ln(S_t) - \ln(S_{t-1}) \]  

Where the log return and is the daily closing price of the index at time \( t \). Table 1 summarises some statistical properties of the log returns. From the table, it is possible to draw some conclusions about the properties of the data. First, the mean is approximately zero and it can be shown that this statistic is not statistically different from it. Secondly, the time series shows a negative skewness and a high kurtosis compared with the Normal distribution, which is 3. Both factors indicate that a Gaussian distribution may not be a good fit to the data.

<table>
<thead>
<tr>
<th>Statistics</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of observations</td>
<td>2,645</td>
</tr>
<tr>
<td>Mean</td>
<td>0.00016</td>
</tr>
<tr>
<td>Median</td>
<td>0.00024</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.13678</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.12096</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.01782</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.03795</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>5.97858</td>
</tr>
</tbody>
</table>
Christoffersen (2012, p.11) proposes the following generic model of asset returns, which is argued to hold the stylized facts provided by Cont (2001):

\[ R_t = \mu_t + \sigma_t \epsilon_t, \text{ with } \epsilon_t \sim i.i.d. D(0,1) \]  

(2)

Where \( \mu_t \) is the mean and \( \sigma_t^2 \) is the variance of the return, both conditioned on past data. The random variable \( \epsilon_t \) is an innovation term, which is assumed to be independent and identically distributed, following the distribution \( D(0,1) \) with mean zero and standard deviation equals to one. Christoffersen further argues it is reasonable to assume that for short time horizons (e.g. daily) the conditional mean is equal to zero. This statement is corroborated by the results of Table 1 above and by an analysis of a confidence interval for the log returns' mean.

Therefore, the standardized returns (i.e. \( \frac{R_t}{\sigma_t} \)) can be plotted in a Q-Q plot against a standard normal distribution to verify whether it follows a \( N(0,1) \). The Q-Q (quantile-quantile) plot is a visual tool to verify whether the empirical quantiles of the data have the same shape as the quantiles of some theoretical distribution. If the log returns are really described by a predefined theoretical distribution, the observations should lie on the line \( y = x \).

Figure 1 shows a Q-Q plot of the empirical standardized returns against a standard normal distribution with mean zero and variance equals to one. The shape of the curve (i.e. inverted "S-shaped") indicates that the more extreme quantiles of the time series are larger than the quantiles of a \( N(0,1) \) and, hence, the assumption of unconditional normally distributed returns is not reasonable (McNeil et al., 2015 p.85).

It is important to mention that in Figure 1, a constant standard deviation is assumed, which means that it is not being considered the time-varying property of the variance and its dependence with past data. These two well-known factors will be analysed in the next section with the fit of a GARCH(1, 1) model.
Besides the Q-Q plot, there are other tests to verify the assumption of normality. Essentially, these numerical tests assess whether the empirical distribution of the data is consistent with a normal distribution (null hypothesis). Therefore, p-values lower than 0.05 will lead to the rejection of the assumption of normally distributed returns at a 5% level of significance. Some of these tests with their respective p-values are shown in Table 2. As they are commonly used in the literature and it is outside the scope of this paper to discuss each test in detail, their formulas will not be shown here. The interested reader can consult the following papers to obtain more useful information about them, Jarque and Bera (1987), Shapiro and Wilk (1965), Anderson and Darling (1954), Smirnov (1948) and Cramér (1928).

Table 2 shows that for daily returns, the data is not consistent with a Gaussian distribution. However, as mentioned above, for longer time intervals, such as monthly returns, we do not have enough evidence against the null hypothesis of normally distributed returns at a 5% level of significance. This is verified by the higher p-values of most of the tests in Table 2. Although the focus of this paper is on the distribution of daily instead of monthly returns, it is important to show the different results obtained using each time interval. Furthermore, one must take into consideration that, when using monthly observations, the lower number of returns may influence the result of such tests.
Table 2 – Shows the p-values for some well-known normality tests applied to daily and monthly returns. The tests were implemented using R packages, such as nortest, tseries and zoo.

<table>
<thead>
<tr>
<th>Test</th>
<th>Daily Returns</th>
<th>p-value</th>
<th>Test</th>
<th>Monthly Returns</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera</td>
<td></td>
<td>0.0000</td>
<td>Jarque-Bera</td>
<td></td>
<td>0.0000</td>
</tr>
<tr>
<td>Shapiro-Wilk</td>
<td></td>
<td>0.0000</td>
<td>Shapiro-Wilk</td>
<td></td>
<td>0.0076</td>
</tr>
<tr>
<td>Kolmogorov-Smirnov</td>
<td></td>
<td>0.0002</td>
<td>Kolmogorov-Smirnov</td>
<td></td>
<td>0.1358</td>
</tr>
<tr>
<td>Anderson-Darling</td>
<td></td>
<td>0.0000</td>
<td>Anderson-Darling</td>
<td></td>
<td>0.3168</td>
</tr>
<tr>
<td>Cramer-von Mises</td>
<td></td>
<td>0.0000</td>
<td>Cramer-von Mises</td>
<td></td>
<td>0.4635</td>
</tr>
</tbody>
</table>

To conclude this section, the assumption of independent returns will be discussed, as proposed by the generic model above, and consequently the presence of volatility clustering in the data. The former will be crucial to the fit of a generalised Pareto distribution for the extreme values.

Ljung and Box (1978) proposes a test to verify the assumption of randomness in the data. That is, it tests the null hypothesis of log returns being a white noise. The p-values of this test for the Ibovespa returns are shown in Table 3. It is noticeable that we do not have enough evidence to reject the null hypothesis of raw returns being i.i.d. at a 5% level of significance; however, the p-value for squared returns shows a strong evidence against white noise hypothesis, indicating that this time series may have nonlinear dependence.

Table 3 – Shows the p-values for the Ljung-Box test applied to raw returns and squared returns. The test was implemented using the R package stats.

<table>
<thead>
<tr>
<th>Type of returns</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raw</td>
<td>0.2312</td>
</tr>
<tr>
<td>Squared</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Curto et al. (2007) argue that in the presence of nonlinear dependence and fat-tailed unconditional distribution, the Lagrange Multiplier test introduced by Engle (1982) can be used as a numerical test to verify whether a time series exhibits conditional heteroskedasticity, also known as autoregressive conditional heteroskedastic (ARCH) effects.
The Lagrange Multiplier tests the absence of ARCH effects in the data as the null hypothesis. Therefore, as shown in Table 4, we have a strong evidence to reject the null hypothesis, suggesting that nonlinearities penetrate the process through the variance (Curto et al., 2009; Hsiech, 1989).

**Table 4** – Shows the p-value of the Lagrange Multiplier test. The test was implemented using the R package FinTS

<table>
<thead>
<tr>
<th>Test</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lagrange Multiplier</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

### 3. Modelling the data

#### 3.1 GARCH model

As defined by Mittnik et al. (1998), volatility clusters can be captured by fitting ARCH or GARCH models, which express the conditional variance as a function of past data (Engle, 1982; Bollerslev, 1986).

Christoffersen (2012, p.70) argues that a simple GARCH(1, 1) model would be given by the following equation:

\[ \sigma_t^2 = \omega + \alpha \sigma_{t-1}^2 + \beta \epsilon_{t-1}^2 \]

(3)

Where \( R_{t-1} \) is given by equation (1), assuming its mean value is equal to zero and the innovation terms, \( \epsilon_i \) are normally distributed with mean zero and variance equals to one. The latter assumption of normality will be relaxed later in this paper. Moreover, the parameters \( \alpha, \beta \) and \( \omega \) will have to be estimated.

As stated by Cont (2001), empirical asset returns show a negative correlation with measures of volatility (i.e. positive returns have a smaller impact on volatility than negative returns of the same magnitude). This is called leverage effect and can be included into a GARCH(1, 1) model as follows:

\[ \sigma_t^2 = \omega + \alpha (R_{t-1} - \theta \sigma_{t-1})^2 + \beta \sigma_{t-1}^2 \]

(4)

It is noticeable that negative returns will have a higher impact on variance than positive returns, if \( \theta > 0 \).
The latter model will be fitted to the data using maximum likelihood estimation. Firstly, increments will be assumed to come from a N(0,1) and other non-normal distributions will be considered later.

3.2 Normal distribution

As mentioned above, a GARCH(1, 1) model with leverage was fitted to the data by using maximum likelihood estimation and their estimates with the respective standard errors are shown in Table 5 below.

Table 5 – Parameters values using maximum likelihood estimation and their respective standard errors, for a GARCH(1, 1) model with leverage and normal innovations

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>0.00000543</td>
<td>0.00000142</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.04673</td>
<td>0.01090</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.92220</td>
<td>0.01295</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.50260</td>
<td>0.13110</td>
</tr>
</tbody>
</table>

Figure 2 shows a Q-Q plot of the empirical standardized quantiles from a GARCH(1, 1) with leverage against the quantiles of a normal distribution. Here we are using time-varying standard deviation, as obtained by the implementation of the model, to calculate standardized returns, instead of using a constant variance as it was used in Figure 1.
As shown in Figure 2, it is noticeable that a GARCH model offers a significant improvement in comparison with the unconditional distribution of daily log returns. Essentially, the GARCH model can capture some of the non-normality inherent to the time series (Christoffersen, 2012, p. 75). Returns from the right tail are captured extraordinarily well by a normal GARCH model. Nonetheless, it does not capture some fat left tail deviations from the $y = x$ line, indicating that the use of normally distributed innovations will underestimate the true probability of large negative returns. This might lead to the underestimation of risk measures, such as the Value-at-Risk and Expected Shortfall, which are primarily concerned with the left tail of the distribution of returns.

Therefore, a conditional distribution that fits better extreme negative returns will be much more useful from a risk management perspective and, hence, other the Student’s $t$ distribution will be used in the next section due to its heavier tails.
3.3 Student’s $t$

The aim of this section is to allow innovation terms to be distributed by a standardized Student’s $t$ distribution, i.e. $\sim t(d)$, where $t(d)$ has mean equals to zero, $d$ degrees of freedom and variance equals to one.

In this paper, we use the quasi-maximum likelihood estimation (QMLE) to estimate the parameters. According to McNeil et al. (2015 p.127) this method assumes that the innovation terms in a GARCH model are incorrectly assumed to be normally distributed, however, the dynamic of the model fitted is correct. McNeil et al. continue defining the properties of this estimation method as if

“...the true data-generating mechanism is a GARCH(p, q) model with non-Gaussian innovations, but we attempt to estimate the parameters of the process by maximizing the likelihood for a GARCH(p, q) model with Gaussian innovations. We still obtain consistent estimators of the model parameters...”.

Hence, in the case of a Student’s $t$ distribution, it is assumed that a normal GARCH model estimates the conditional variance and we just need to estimate the degrees of freedom of the distribution by using maximum likelihood.

Figure 3 – Q-Q plot of the empirical standardized returns from a GARCH(1, 1) model with leverage against the theoretical quantiles of a Student's t distribution with 8.85 degrees of freedom
The estimated $d$ parameter of the $t$ distribution is 8.85 and a Q-Q plot of the standardized returns from a GARCH(1, 1) model with leverage against a theoretical Student’s $t$ distribution is shown on Figure 3 above. It is noticeable that this distribution has a better fit than a normal distribution for the left tail, although there are some extreme negative returns that deviate quite considerably from the 45-degree line. Additionally, due to the symmetry of this distribution, the left tail starts to deviate from the $y = x$ line, indicating that an asymmetric distribution may be more reasonable. This result is consistent with the work of Christoffersen (2012 p.132) who has used the same approach described above to analyse the daily returns of the S&P 500.

3.4 Extreme Value Theory

McNeil et al. (2015 p.135) define Extreme value theory (EVT) as “…a branch of probability concerned with limiting laws for extreme values in large samples.”.

Essentially, by fitting an EVT distribution, such as a generalised Pareto distribution (GPD), we are trying to model the behaviour of extreme values (left and right tail) separately. This allows us to avoid assumptions about the symmetry of the empirical data, as it was the case with a Student’s $t$ distribution when the returns on the extreme right side of the distribution deviated from the $y = x$ line (see Figure 3). Therefore, a threshold must be defined such that the data above (below) this positive (negative) value will be modelled according to the specified distribution.

The threshold was chosen by using the Q-Q plot shown in Figure 2 as a visual guidance (see McNeil and Frey, 2000; McNeil and Saladin, 1997).

Christoffersen (2012 p.138) argues that one of the main results in EVT asserts that as the threshold $u$ go to infinity, the distribution of observations over the threshold is a generalised Pareto distribution and its cumulative density function is given by:

$$ GPD(x; \xi, \beta) = \begin{cases} 1 - \left(1 + \frac{\xi x}{\beta}\right)^{-\frac{1}{\xi}} & \text{if } \xi \neq 0 \\ 1 - \exp\left(-\frac{x}{\beta}\right) & \text{if } \xi = 0 \end{cases} $$

Where $\beta > 0$, $x \geq u$ and $\xi$ is the tail index parameter. The latter is commonly assumed to be greater than zero in risk management literature, indicating heavy tails in the distribution (see McNeil, 1999). The same is assumed in this paper and the Hill estimator will be used to estimate the tail index parameter (see Hill, 1975).
Additionally, Christoffersen, (2012 p.138) argues that an assumption of independent and identically distributed returns is extremely important to the correct fit of a GPD distribution. Hence, the normal GARCH(1, 1) model with leverage is fitted to the time series, in order to deal with the time-varying property of the variance. After this, extreme value theory is applied to the standardized returns (i.e. it is considered that after applying the GARCH model, the assumption of i.i.d. standardized returns is reasonable and one can use EVT on them).

**Figure 4** – Q-Q plot of the empirical quantiles from a GARCH(1, 1) model with leverage against the theoretical quantiles of a generalised Pareto distribution

![Image](https://example.com/figure4.png)

Figure 4 shows a Q-Q plot of the empirical quantiles of the largest positive and negative returns against the quantiles of the GPD distribution. In this figure it is easy to verify what we meant by saying that both tails should be modelled separately. Left and right tail have a different tail index estimate and, hence, we do not need to rely on the assumption of symmetry in the data. Moreover, it is noticeable that an EVT methodology fits the extremes better than the previous distributions discussed in this paper.

Next section will discuss the impact on the Value-at-Risk and the Expected Shortfall of using different distributions to model daily returns.
4. Calculating risk measures

4.1 Value-at-Risk (VaR)

VaR is a commonly used risk measure simply defined as the $100\alpha - \text{th}$ quantile of the log returns distribution or, similarly, the $100(1 - \alpha) - \text{th}$ quantile of the loss distribution over a fixed time horizon. That is, we are $(1 - \alpha)%$ confident that we will get returns greater than the VaR.

Thus, assuming is the random variable denoting the returns until time $t$, the VaR with $(1 - \alpha)%$ confidence level is defined as

$$\Pr (R_t < - VaR^\alpha_t) = \alpha$$

(6)

Where the negative sign is needed because the Value-at-Risk is defined as a positive number. Therefore, is the value so that we will only observe returns below it with probability $\alpha$. In this paper, a 99% confidence level (i.e. $\alpha = 0.01$) is considered.

Figure 5 shows the daily VaR for all conditional distributions considered in section 3 from January 2017 to May 2017. It is noticeable that the Gaussian distribution produces lower values than the other two distributions. Essentially, considering the results of goodness of fit obtained in the previous section, it means that a risk manager using the assumption of normality will consistently underestimate the Value-at-Risk of his financial position.

Figure 5 – Daily VaRs for the conditional distributions with a 99% confidence level, from January 2017 to April 2017
Value-at-risk, by its very definition, is a quantile and, hence, it does not give any additional information about negative returns with probability lower than $\alpha$ and how large they can be. Moreover, Artzner et al. (1999) argue that the VaR measure does not obey all the properties of a coherent risk measure (i.e. homogeneity, subadditivity, monotonicity and translation invariance) and they showed that Expected Shortfall does.

4.2 Expected Shortfall (ES)

It is important for a risk manager to consider how heavy a loss can be. As discussed above, VaR is not concerned with the magnitude of these losses and, hence, Expected Shortfall is sometimes used to overcome this drawback.

Christoffersen, (2012 p.33) argues that for a continuous distribution of returns $R_t$, and for a given $\alpha$, the ES can be defined as follow:

$$ES_t^\alpha = -E_{t-1}[R_t | R_t < -VaR_t^\alpha]$$  \hspace{1cm} (7)

Here again, the negative sign is necessary because the ES is defined as a positive number. Expected Shortfall is described as the expected return given that the return is lower than the Value-at-Risk (Christoffersen, 2012 p.126). Therefore, the ES reflects the full left tail of the distribution.

Figure 6 shows daily ES for all conditional distributions considered in section 3 from January 2017 to May 2017, with 99% confidence level. Firstly, it is noticeable the upper shift of the curves for all distributions, in comparison with Figure 5, indicates that given that the returns are below the Value-at-Risk, the expected return might be considerably lower. This is important from a risk management perspective because such loss may completely jeopardize the financial position or can cause a bankruptcy of a company.
Most importantly, Figure 6 shows how heavier a loss might be if the underlying conditional distribution of extremes follows a generalised Pareto distribution with $\alpha = 0.01$. That is, a risk manager following the assumption of normally distributed returns and using Expected Shortfall as a risk measure may be considerably underestimating how huge a loss can be.

It is important to mention that as we decrease the value of $\alpha$ (i.e. increasing the confidence level to 99.5% or 99.9%, for example) the values of daily ES increase. However, the proportion of this increase affects each distribution differently. For example, with $\alpha = 0.01$ the average values of ES for the period between January 2017 and April 2017 are 3.58%, 3.84% and 4.36% for Normal, Student’s $t$ and GPD distribution, respectively. Nevertheless, with $\alpha = 0.001$ these values go to 4.52%, 6.65% and 7.63%. This is due to the longer tails of Student’s $t$ and GPD distribution in comparison with a normal distribution. A similar statement is valid to the Value-at-Risk.
5. Conclusion

The results obtained after analysing the Ibovespa daily returns show that the normal distribution is a poor fit for the unconditional distribution. This is due to the fat tails of the empirical data. Moreover, the time series exhibits some nonlinear dependence that might be captured by a GARCH model.

The fit of a GARCH(1, 1) model with leverage using Normal and Student’s t innovations show that the latter is a better fit to the right tail of the conditional distribution, which is essentially important from a risk management perspective. In addition to that, the fit of GPD shows that this distribution is a better fit for both tails of the empirical data.

The use of different distributions as models of asset returns will have a large impact on the risk measures used to assess the risk of a company’s financial position. Therefore, this paper concludes showing how each conditional distribution have different measures of Value-at-Risk and Expected Shortfall. Essentially, both GPD and the Student’s t distribution have a bigger VaR and ES than the normal. However, the generalised Pareto distribution has a much bigger ES in comparison with the other two distributions.

6. Bibliographical References


